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Laumon sheaf and the mod p Langlands program for GL_2 of a finite degree extension of $\mathbb{Q}_p$

$E|\mathbb{Q}_p$

$d = [E : \mathbb{Q}_p]$

$\rho \in \mathrm{Irr}_{\overline{\mathbb{F}}_q}(\mathrm{Gal}(\overline{E}|E))$

$$\pi(\rho)\Big|_{\begin{pmatrix} E^\times & E \\ 0 & E^\times \end{pmatrix}} = \mathcal{F}\left(j_!\, RH\left(S_d \mathcal{F}_\rho\right)\right)$$

- S_d: Symmetrization
- $\mathcal{F}_\rho$: étale loc. sys. associated to ρ
- RH: Riemann-Hilbert
- $j_!$: extension by zero as a Ш-q.c. sheaf "à la Mann"
- $\mathcal{F}$: Fourier transform Mukai type

Philosophy MSRI 2014

(p, ℓ) — $\boxed{\ell \neq p}$. F.-Scholze. Diamonds.

Space ↗ p, coefficients ↗ ℓ

(∞, ∞) Scholze - Liquid.

Today: (p, p) p-adic Langlands.

A lot is known for $GL_2(\mathbb{Q}_p)$ (Breuil, Colmez, Emerton, Kisin, Paskunas ...). Everything breaks down for $GL_2(E)$, $E \neq \mathbb{Q}_p$

$(p, p) \longleftrightarrow (p, \ell)$ know how to do "more or less"

Key word today: $\boxed{\text{Holonomy}}$.

Warm up: Recall (Laumon): X/k smooth projective curve

$\mathcal{E}/X$ irreducible $\overline{\mathbb{Q}}_\ell$-loc. system.

$d \geq 1$, $\pi_d : X^d \longrightarrow \mathrm{Div}^d_X = X^d/\sigma_d$.

$$S_d \mathcal{E} := \left[\pi_{d*} \mathcal{E}^{\boxtimes d} \right]^{\sigma_d} \in \mathrm{Perv}(\mathrm{Div}^d_X, \overline{\mathbb{Q}_\ell})$$

$\|$

$j_{!*} \mathcal{F}_d$ where if $U_d \subset X^d$

$U_d = \{x_1, \dots, x_d \mid \forall i \neq j,\ x_i \neq x_j\}$

$$\begin{array}{c} U_d \\ \downarrow \big) \sigma_d \\ \pi_d(U_d) \xrightarrow{j} \mathrm{Div}^d_X \end{array}$$

$\mathcal{F}_d$ ← loc. sys. associated to σ_d-equivariant loc. sys. $\mathcal{E}^{\boxtimes d}|_{U_d}$. [Analog wt. Springer sheaf]

* $\mathcal{E}$ = vector bundle / X - $S_d \mathcal{E} := \left[\pi_{d*} \mathcal{E}^{\boxtimes d} \right]^{\sigma_d}$ = Coherent sheaf on Div^d_X

[[Lemma. $S_d \mathcal{E}$ = vector bundle.]]

Stratification by multiplicity of a divisor

$\underbrace{\mathrm{Div}^{d,\lambda}_X}_{\text{locally closed}}$ $\quad \lambda \in \Lambda_d = \{\text{partitions of } d\}$

$\underline{\lambda} = (\lambda_1, \dots, \lambda_n)$ $\quad \sum_i \lambda_i = d.$

$\mathfrak{S}_{\underline{\lambda}} = \{ \sigma \in \mathfrak{S}_n \mid \forall i, \lambda_{\sigma(i)} = \lambda_i \}$

$$\begin{array}{ccc} U_n \subset X^n & & (x_1, \dots, x_n) \\ \big\downarrow \ \mathfrak{S}_{\underline{\lambda}}\text{ - Galois cover} & & \big\downarrow \\ \mathrm{Div}_X^{d,\underline{\lambda}} \xrightarrow{i^{\underline{\lambda}}} \mathrm{Div}_X^d & & \sum_i \lambda_i [x_i] \end{array}$$

[Prop: d invertible in k. $(i^{\underline{\lambda}})^* S_d \mathcal{E}$ = v. b. associated to the $\mathfrak{S}_{\underline{\lambda}}$ - equivariant v. b. $(T_{\lambda_1}\mathcal{E} \boxtimes \dots \boxtimes T_{\lambda_n}\mathcal{E})_{|U_n}$]

where $T_k(-)$ = polynomial functor that is a twisted form of $(-)^{\otimes k}$

Compatibility wrt. R. H. $\quad X/\mathbb{F}_q$ scheme

Recall: Katz: $\{\mathbb{F}_q\text{-étale local systems}\} \xrightarrow{\sim} \{(\mathcal{E}, \varphi) \mid \mathcal{E} = \text{v. b. on } X, \ \varphi : \mathrm{Frob}_q^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}\}$

$$\mathcal{F} \longmapsto (\mathcal{F} \otimes_{\mathbb{F}_q} \mathcal{O}, \mathrm{Id} \otimes \mathrm{Frob}_q)$$

$$\mathrm{Sol}(\mathcal{E}, \varphi) := \tilde{\mathcal{E}}^{\varphi = \mathrm{Id}} \longleftarrow (\mathcal{E}, \varphi)$$

Emerton-Kisin, Bhatt-Lurie.

RH: $\{\text{étale sheaves of } \mathbb{F}_q\text{-v.s.} / X\} \hookrightarrow \left\{ (\mathcal{E}, \varphi) \;\middle|\; \begin{array}{l} \mathcal{E} = \text{q.c. on } X \\ \varphi : \mathcal{E} \xrightarrow{\sim} \mathcal{E} \\ \text{semi-linear} \end{array} \right\}$

- Extends Katz correspondence -

Prop.: $\mathcal{F} = \mathbb{F}_q$ étale loc. sys. $/X$ (curve now) - ~~$\mathcal{E}$ Katz($\mathcal{F}$)~~

$\Rightarrow$ $RH(S_d \mathcal{F}) = \varinjlim_{\varphi} \{ S_d \mathcal{E},$ (v.b.) $\qquad (\mathcal{E}, \varphi) = \text{Katz}(\mathcal{F})$.

(perfection)

<u>The real thing</u>: $E | \mathbb{Q}_p \quad d = [E : \mathbb{Q}_p]$.

$\mathbb{B}^{\varphi = \pi}$ = absolute B.C. space $\simeq \mathrm{Spd}(\bar{\mathbb{F}}_q [[T^{1/p^\infty}]])$

$\downarrow$

$* = \mathrm{Spd}(\bar{\mathbb{F}}_q)$

$\mathrm{Rep}_{\bar{\mathbb{F}}_q}(\Gamma_E) \simeq$ Étale loc. syst. on $\underbrace{\mathbb{B}^{\varphi=\pi} \setminus \{0\} / E^\times}_{\mathrm{Div}^1}$

RH: $\mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \xrightarrow{\sim} \{\text{vector bundles on } \mathrm{Div}^1\}$

$$\rho \longmapsto \underbrace{\mathcal{F}_\rho \otimes \mathcal{O}}$$

Lubin-Tate (φ, Γ)-module associated to ρ.

$\mathcal{E}_\rho := \mathcal{F}_\rho \otimes \mathcal{O}$ seen as an $\underline{E}^\times$-equivariant v.b. on $\mathbb{B}^{\varphi=\pi} \setminus \{0\}$

$$\pi_d : \left(\mathbb{B}^{\varphi=\pi} \setminus \{0\}\right)^d \longrightarrow \mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$$

$$(x_1, \dots, x_d) \longmapsto x_1 \cdots x_d$$

$\underline{\mathrm{Th}}$ (F.): $\Delta_d = \{(\mu_1, \dots, \mu_d) \in (E^\times)^d \,/\, \prod_{i=1}^{d} \mu_i = 1\}$

π_d is quasi-pro-étale surjective inducing

$$\left(\mathbb{B}^{\varphi=\pi} \setminus \{0\}\right)^d \Big/ \underline{\Delta_d \rtimes \mathfrak{S}_d} \xrightarrow{\sim} \mathbb{B}^{\varphi=\pi^d} \setminus \{0\}$$

↖ ~~[illegible]~~-pro-étale quotient

Def. $S_d \mathcal{F}_\rho = \left[\pi_{d*} \mathcal{F}_\rho^{\boxtimes d}\right]^{\Delta_d \rtimes \mathfrak{S}_d}$ ← étale sheaf

$S_d \mathcal{E}_\rho = \left[\pi_{d*} \mathcal{E}_\rho^{\boxtimes d}\right]^{\Delta_d \rtimes \mathfrak{S}_d}$ ← sheaf of $\mathcal{O}$-modules

$B^{\varphi=\pi^d} \setminus \{0\} = \mathrm{Spa}\left(\overline{\mathbb{F}}_q[[x_1^{1/p^\infty}, \dots, x_d^{1/p^\infty}]], -\right) \setminus V(x_1, \dots, x_d)$

= q.c. perfectoid space

$S_d : \mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \xrightarrow{\text{degree } d} \left\{ \begin{array}{l} \text{overconvergent étale sheaves of} \\ \overline{\mathbb{F}}_q\text{-v.s. on } \mathrm{Spa}(\overline{\mathbb{F}}_q[[x_1, \dots, x_d]], -) \setminus V(x_1, \dots, x_d) \end{array} \right\}$

polynomial functor

(the last space is a Noetherian adic space)

→ Can describe $(i^\lambda)^* S_d \mathcal{E}_\rho$ as in the "classical case".

Conjecture (Holonomicity Conjecture):

$S_d \mathcal{E}_\rho$ = Completion of the perfection of an $\mathcal{O}$-module that is a perfect complex.

→ O.K. if we replace E by $E = \mathbb{F}_q((\pi))$ (equal-char. case)

Th: Holonomy Conjecture ⇒ $\mathrm{Sd}\,\mathcal{E}_\rho$ generated by its global sections.

Point: $\overline{\mathbb{F}}_q[[E]] = \mathcal{O}(\mathbb{B}^{\varphi=\pi^d})$

G

$E^\times$ $\begin{pmatrix} E^\times & E \\ 0 & 1 \end{pmatrix}$

global sections of $\mathcal{E}_\rho$ defines a functor

$$\mathrm{Rep}_{\overline{\mathbb{F}}_q}(\Gamma_E) \longrightarrow \mathrm{Rep}_{\overline{\mathbb{F}}_q}\begin{pmatrix} E^\times & E \\ 0 & 1 \end{pmatrix}.$$

Expected to be ~~[illegible]~~ the mod p Langlands for $GL_2(E)$.